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# STRATEGIC SEARCH THEORY

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## ABSTRACT

This paper combines the ''theory of search'' -- the application of optimal stopping rules to decision-making under uncertainty -- with concepts from the theory of games in order to analyze new product development. A development trial is envisioned as a random drawing of a production cost level, and a strategy is a rule describing conditions under which no further development is desired -- a stopping rule. Nash equilibrium in stopping rules is shown to exist and possess the reservation property. The possibility of multiple equilibria implies that the usual comparative statics results need not hold in equilibrium -- e.g., an increase in firm i's development costs may result in an increase in the firm's development activity.

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## I. INTRODUCTION

The ''theory of search'' -- the application of optimal stopping rules to decision-making under uncertainty -- is well-developed for the case of an individual agent [Kohn and Shavell, 1974; Lippman and McCall, 1976; Rothschild, 1973]. It has been applied primarily to the problems of job search and consumers searching for the lowest price. However, a number of extensions, generalizations and new applications have been offered recently. Among these are the dynamic analysis of the inspection and evaluation of multiple-characteristic goods [Wilde, 1980]; the characterization of optimal search when sampling from a number of different distributions [Spulber, 1979; Weitzman, 1979]; and the application of standard search theory to the determination of market structure via uncertain imitability [Lippman and Rumelt, 1980], research and development [Evenson and Kislev, 1976; Spulber, forthcoming; Lee, 1980], and the decision to adopt a new technology of uncertain value [Jensen, 1979].

In job search and consumer search applications, it is

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reasonable to treat the decision-maker's problem as independent of the behavior of other searchers. But in the industrial organization problems mentioned above, one is interested in the effects of the behavior of rival firms upon a firm's decision-making process. This paper combines search theory with concepts from the theory of games in order to analyze new product development.

In Section II, the one-firm case is reviewed and related to other current research. In Section III the existence of Nash equilibrium in stopping rules is demonstrated. The equilibrium stopping rules are shown to possess the reservation property. Section IV discusses comparative statics results, and Section V examines the effects of rivalry. Section VI concludes and suggests extensions.

## II. THE ONE-FIRM CASE

The idea of treating development activity as sampling from a distribution of economic returns is due to Evenson and Kislev [1976]. Spulber [forthcoming] and Lee [1980] separate research and development into two distinct but interrelated activities. Development is treated as a problem in sequential search, parametrized by the level of a technological index generated by prior research activity. A similar idea is found in Lippman and Rumelt's [1980] concept of uncertain imitability. A firm considering entry into an industry must draw its production cost level from a distribution of possible costs; it cannot perfectly imitate the technology of other firms. Neither of these



perfectly imitate the technology of other firms. Neither of these
treatments is strategic -- i.e., game theoretic.[1] Thus the modeling of development as a problem in sequential search is not new -- it is the addition of the strategic element which distinguishes this analysis from those discussed above.

Assume that basic research, which is common knowledge, has made it apparent that a particular new product is feasible and limits the range of (constant unit) production costs to an interval $[\underline{c},\bar{c}]$. Any of a number of potential producers may produce the new product at a cost of $\bar{c}$ per unit. However, by expending resources on development activity, a firm may be able to reduce production costs below $\bar{c}$. The production cost parameter, c, which results from a development trial, is assumed to be a random variable representable as a draw from a probability distribution defined over $[\underline{c},\bar{c}]$. Development costs (per trial) are assumed fixed. Sampling is assumed to be essentially instantaneous, occurring before the market officially opens. Demand for the product is assumed stationary. Therefore, maxmized profits are representable in present value terms, dependent only on production costs.

Consider the single-firm problem. If the firm produces at cost level c, it receives the corresponding level of monopoly profit, denoted by $\pi(c)$. Given the assumptions made so far,

$$\pi(c) \equiv \max_{q} \frac{1}{r} \{p(q)q - cq\}.$$

Profits are assumed to obey the following restrictions.

A1. $\pi(c)$ is non-negative, bounded and continuously differentiable with $\pi'(c) < 0$ on $[\underline{c},\bar{c}]$.

Development activity is assumed to consist of drawing a production cost level from $[\underline{c},\bar{c}]$ according to the continuously differentiable distribution function $F(c)$. Let $X_1, X_2, \ldots, X_n$ represent a sequence of such independent draws (development trials), and let $k$ represent the fixed level of development costs per trial. Define

$$Y_n = \max\{\pi(x_1), \ldots, \pi(x_n)\} - nk.$$

<u>Definition 1.</u> A nonnegative integer-valued random variable N is said to be a <u>stopping rule</u> for the sequence $X_1, \ldots, X_n$ if the occurrence or nonoccurrence of the event $\{N = n\}$ can be determined by looking at the values of $X_1, \ldots, X_n$ (Cinlar, 1975).

The firm wants to choose a stopping rule so as to maximize its expected gain $E[Y_N]$, where N is the (random) stopping time.

An argument due to Robbins [1970] guarantees the existence of an optimal stopping rule which is myopic so long as the random variables $\pi(X_1), \pi(X_2), \ldots,$ are independent and identically distributed (i.i.d.) with finite mean. Since $X_1, X_2, \ldots$ are i.i.d and $\pi(c)$ is bounded, both requirements are met. Thus the optimal stopping rule is of the form

stop whenever $E[Y_{n+1} \mid x_1, \ldots, x_n] \leq y_n$;
otherwise continue.



But

$$E[Y_{n+1} \mid x_1, \ldots, x_n] = y_n + \int_{\underline{c}}^{m} [\pi(c) - \pi(m)]\,dF(c) - k$$

where $m = \min\{x_1, \ldots, x_n\}$. Thus the condition for stopping is: stop if and only if

$$V(m) = \int_{\underline{c}}^{m} [\pi(c) - \pi(m)]\,dF(c) - k \leq 0. \tag{1}$$

$V(m)$ represents the value of one additional draw when the best draw to date is m. The rule is to stop whenever an additional draw has non-positive value. Since the firm can elect to forego development and produce at $\bar{c}$, the sign of $V(\bar{c})$ is important.

A2. (a) $V(\bar{c}) > 0$.
(b) $V(\bar{c}) \leq 0$.

If A2(a) holds, then $V(\bar{c}) > 0$, $V(\underline{c}) < 0$ and because $V(m)$ is continuously differentiable with

$$V'(m) = \int_{\underline{c}}^{m} - \pi'(m)\,dF(c) = -\pi'(m)F(m) > 0,$$

there exists a unique $\xi \in (\underline{c},\bar{c})$ such that $V(m) \gtrless 0$ as $m \gtrless \xi$. If A2(b) holds, then no development is optimal; $\xi \equiv \bar{c}$.

Thus for the model of Section II, the optimal stopping rule can be characterized in terms of the reservation cost level $\xi$:

stop whenever $m \leq \xi$;

otherwise continue development.

It is easy to show that (when A2(a) holds) $\xi$ increases with development costs $k$; that is, as development costs increase, so does the threshold level below which development ceases. Furthermore, a mean profit-preserving spread in the distribution $F(c)$ [Diamond and Stiglitz, 1974] results in a lower reservation cost level [see, e.g., Kohn and Shavell, 1974].

## III. STRATEGIC SEARCH EQUILIBRIUM

If a number of firms are undertaking development, then the value (to any one firm) of developing the product depends not only on its own production costs but also on the production costs of its rivals. This is because ultimately the firms will supply the new product oligopolistically and therefore price and market shares will depend upon the production costs of all firms. Thus a generalization of the search model which allows game-playing rivals is required.

This paper focuses on the two-person case so as to highlight the conceptual differences between the game model and that of Section II without the technical complexity of the n-person case.

Suppose that there are two potential suppliers of the new good, named 1 and 2. Let $c_i$ represent firm i's constant marginal production



cost. Then $\pi_i(c_1,c_2)$ denotes the present value of i's profits, generated in a game of duopoly in the new product market.[2]

A3. $\pi_i(c_1,c_2)$ is assumed to be bounded, non-negative and twice differentiable with $\frac{\partial \pi_i}{\partial c_i}(c_1,c_2) < 0$, $\frac{\partial \pi_i}{\partial c_j}(c_1,c_2) > 0$, and $\frac{\partial^2 \pi_i}{\partial c_i \partial c_j}(c_1,c_2) < 0$ for all $(c_1,c_2) \in [\underline{c},\bar{c}]^2$.

That is, as the rival's cost parameter increases, i's profits increase, but less so as i's own production cost parameter increases.[3]

In this section the problem will be formulated in a standard game theoretic framework. Accordingly, each firm will possess full information regarding the opponent's strategy space and payoff but will not be able to observe the outcomes of its rival's development trials.

One needs to define random variables analogous to those of Section II. Let $X_1^i, X_2^i, \ldots, X_{n_i}^i$ represent a sequence of independent random variables (or development trials) for firm i. Each is drawn from $[\underline{c},\bar{c}]$ according to the continuously differentiable distribution function $F_i(c_i)$. Let $k_i$ be firm i's fixed level of development costs per trial and $m_i = \min\{x_1^i, \ldots, x_{n_i}^i\}$. Firm i is assumed to know the results of its own development trials, but is <u>unable</u> to observe the results of its rival's trials.

<u>Definition 2.</u> A <u>strategy</u> for i is a stopping rule for the sequence

$X_1^i, X_2^i, \ldots, X_{n_i}^i$, and will be denoted $N_i$. The strategy space for i is the space of all such stopping rules.

Since firm i does not know the outcomes of its rival's development trials, the actual stopping cost $m_j^{N_j} = \min\{x_1^j, \ldots, x_{N_j}^j\}$ is a random variable to i. However, given any stopping rule $N_j$, i can compute $\Pr\{m_j^{N_j} \leq c\}$.

Define

$$Y_{n_1}^1(N_2) = \{E[\pi_1(x_1^1, m_2^{N_2})], \ldots, E[\pi_1(x_{n_1}^1, m_2^{N_2})]\} - n_1 k_1.$$

$Y_{n_2}^2(N_1)$ is similarly defined.

The objective of firm i is to choose a stopping rule $N_i$ so as to maximize its expected gain, taking j's strategy $N_j$ as given. That is, i wants to choose $N_i$ so as to maximize

$$E[Y_{N_i}^i(N_j)].$$

Definition 3. A strategy $N_i$ is a best response for i to $N_j$ if $N_i$ maximizes $E[Y_{N_i}^i(N_j)]$.

Proposition 1. A best response for i to $N_j$ exists and is myopic. That is, the best response is of the form

stop whenever $E[Y_{n_i+1}^i(N_j) | x_1^i, \ldots, x_{n_i}^i] \leq y_{n_i}^i(N_j)$;



otherwise continue.

Proof: For i = 1. The proof for i = 2 is analogous. For any $N_2$, this problem is isomorphic to that of Section II, where it is now required that $E[\pi_1(x_1^1, m_2^{N_2})], \ldots, E[\pi_1(x_{n_1}^1, m_2^{N_2})]$ are i.i.d. with finite mean. Since $X_1^1, \ldots, X_{n_1}^1$ are i.i.d. and $\pi_1(c_1, c_2)$ is bounded, both requirements are met. Therefore, for any $N_j$, Robbins' [1970] result again guarantees existence of an optimal stopping rule which is myopic.

Q.E.D.

Proposition 1 states that a myopic stopping rule (sequential search behavior) is a best response to any rival stopping rule—even one involving nonsequential search. Thus, if a Nash equilibrium exists, it will involve sequential search behavior. The best response is further characterized below.

For given $N_2$,

$$E[Y_{n_1+1}^1(N_2) | x_1^1, \ldots, x_{n_1}^1] = y_{n_1}^1(N_2) + E[V^1(m_1, m_2^{N_2})]$$

where $V^1(m_1, m_2^s) \equiv \int_{\underline{c}}^{m_1} [\pi_1(c_1, m_2^s) - \pi_1(m_1, m_2^s)] dF_1(c_1) - k_1$ is the value to 1 of an additional draw when $m_1$ is player 1's best draw to date and 2 stops with cost parameter $m_2^{N_2}$. Thus the condition for stopping (for 1) is: stop whenever

$$U^1(m_1,N_2) \quad E[\int_{\underline{c}}^{m_1} [\pi_1(c_1,m_2^{N_2}) - \pi_1(m_1,m_2^{N_2})]dF_1(c_1) - k_1] \leq 0.$$

Proposition 2. The best response for 1 to $N_2$ can be characterized by a reservation cost level $\phi_1(N_2) \in [\underline{c},\bar{c}]$. That is, the best response is to

stop whenever $m_1 \leq \phi_1(N_2)$;
otherwise continue.

Proof: Recall that $N_2$ is independent of $m_1$ since $m_1$ is unobservable to 2. Then $U^1(m_1,N_2)$ is differentiable in $m_1$ with $\partial U^1/\partial m_1 = E[-(\partial\pi_1(m_1,m_2^{N_2})/\partial c_1)F_1(m_1)] > 0$ since $\partial\pi_1/\partial c_1 < 0$ for all $(c_1,c_2)$. Note that $U^1(\underline{c},N_2) = -k_1 < 0$ for all $N_2$. If $U^1(\bar{c},N_2) > 0$, then by the continuity and monotonicity of $U^1$ in $m_1$, there exists a unique number $\phi_1(N_2) \in (\underline{c},\bar{c})$ such that $U^1(m_1,N_2) \gtrless 0$ as $m_1 \gtrless \phi_1(N_2)$. If $U^1(\bar{c},N_2) \leq 0$, then define $\phi_1(N_2) \equiv \bar{c}$. In either case, if it is optimal to stop at $m_1 = \phi_1(N_2)$, then it is optimal to stop for all $m_1 < \phi_1(N_2)$ as well. Therefore the best response can be characterized as

stop whenever $m_1 \leq \phi_1(N_2)$;
otherwise continue.

Q.E.D

Thus the stopping set for 1 is a connected set $[\underline{c},\phi_1(N_2)]$ for any $N_2$. Hence if a Nash equilibrium exists, it is in strategies with the reservation property. Propositions 1 and 2 imply that one may eschew the space of arbitrary stopping rules and, without loss of generality,



restrict consideration to myopic stopping rules with the reservation property. Thus one can redefine the notions of strategy, payoff and best response in terms of this restricted space of stopping rules. This is done in the remainder of the paper. The same notation will be used, with the understanding that $\xi_j$ will be substituted for $N_j$.

Definition 2'. A strategy for i is a stopping rule of the form

stop whenever $m_i \leq \xi_i$;
otherwise continue.

The strategy space for i is the space of all such stopping rules. Since these rules can be indexed by $c \in [\underline{c},\bar{c}]$, one could equivalently define the strategy space for i to be $C_i = [\underline{c},\bar{c}]$, $i = 1,2$.

Since firm i does not know the outcomes of its rival's development trials, the actual stopping cost $m_j^{N_j}$ is a random variable to i. However, given any strategy $\xi_j \in C_j$, i knows that $m_j^{N_j} \in [\underline{c},\xi_j]$. Furthermore, $\Pr\{m_j^{N_j} \leq c_j\} = F_j(c_j)/F_j(\xi_j)$ for all $c_j \in [\underline{c},\xi_j]$. To see this, simply note that j stops the first time $m_j$ enters $[\underline{c},\xi_j]$. The distribution of the "failures," $x_{n_j}^j > \xi_j$, is irrelevant. The distribution of the successful one, given that it is successful, is just $F_j(c_j)/F_j(\xi_j)$.

If $\xi_j = \bar{c}$, then j undertakes no search. Thus $m_j^{N_j} = \bar{c}$ with probability one. Firm j simply exercises its option to produce at the highest known feasible cost $\bar{c}$.

Define

$$Y^1_{n_1}(\xi_2) = \max\left\{E[\pi_1(x^1_1,m_2^{N_2})],\ldots,E[\pi_1(x^1_{n_1},m_2^{N_2})]\right\} - n_1k_1$$

$$= \begin{cases} \int_{\underline{c}}^{\xi_2} \pi_1(m_1,c_2)dF_2(c_2)/F_2(\xi_2) - n_1k_1 & \xi_2 < \bar{c} \\ \pi_1(m_1,\bar{c}) - n_1k_1 & \xi_2 = \bar{c}. \end{cases}$$

$Y^2_{n_2}(\xi_1)$ is similarly defined.

Then the objective of firm i is to choose a stopping rule so as to maximize its expected gain, <u>taking j's strategy</u> (with reservation cost $\xi_j$) <u>as given</u>. That is, i wants to maximize

$$E[Y^i_{N_i}(\xi_j)],$$

where $N_i$ is i's (random) stopping time.

Proposition 1 states that optimal behavior for firm i requires the use of a myopic stopping rule.

For all $\xi_2 \in [\underline{c},\bar{c})$,

$$E[Y^1_{n_1+1}(\xi_2)|x^1_1,\ldots,x^1_{n_1}] =$$

$$y^1_{n_1}(\xi_2) + \int_{\underline{c}}^{m_1}\int_{\underline{c}}^{\xi_2} [\pi_1(c_1,c_2) - \pi_1(m_1,c_2)]dF_2(c_2)dF_1(c_1)/F_2(\xi_2) - k_1$$



$$= y^1_{n_1}(\xi_2) + \int_{\underline{c}}^{\xi_2} V^1(m_1,c_2)dF_2(c_2)/F_2(\xi_2),$$

where $V^1(m_1,c_2) \equiv \int_{\underline{c}}^{m_1} [\pi_1(c_1,c_2) - \pi_1(m_1,c_2)]dF_1(c_1) - k_1$ is the value to 1 of an additional draw when $m_1$ is player 1's best draw to date and 2 stops with cost parameter $c_2$.

Thus the condition for stopping (for 1) is

$$U^1(m_1,\xi_2) \equiv \int_{\underline{c}}^{\xi_2} V^1(m_1,c_2)dF_2(c_2)/F_2(\xi_2) \leq 0. \tag{2}$$

Notice that now firm 1 decides to stop (or continue) based on the <u>expected</u> value of an additional draw when 1's best draw to date is $m_1$, and given that 2 plays the strategy $\xi_2 \in [\underline{c},\bar{c})$.

If player 2 plays $\xi_2 = \bar{c}$, since this implies <u>no search</u> by 2,

$$E[Y^1_{n_1+1}(\bar{c})|x^1_1,\ldots,x^1_{n_1}] = y^1_{n_1}(\bar{c}) + V^1(m_1,\bar{c}).$$

Therefore $U^1(m_1,\xi_2)$ is discontinuous at $\xi_2 = \bar{c}$, and at this point, firm 1 should stop whenever

$$U^1(m_1,\bar{c}) \equiv V^1(m_1,\bar{c}) \leq 0. \tag{3}$$

Similarly, for firm 2, the optimal rule is to stop whenever

$$U^2(\xi_1,m_2) \equiv \int_{\underline{c}}^{\xi_1} V^2(c_1,m_2)dF_1(c_1)/F_1(\xi_1) \leq 0 \tag{4}$$

for $\xi_1 \in [\underline{c},\bar{c})$, and whenever

$$U^2(\bar{c},m_2) \equiv V^2(\bar{c},m_2) \leq 0. \tag{5}$$

It is clear that the expressions $V^1(m_1,c_2)$ and $V^2(c_1,m_2)$ will be important in what follows.

<u>Proposition 3.</u> $\partial V^i/\partial m_i > 0$ and $\partial V^i/\partial c_j > 0$ for all $m_i \in [\underline{c},\bar{c}]$ and for all $c_j \in [\underline{c},\bar{c}]$.

<u>Proof</u>. For $i = 1$, $j = 2$.

$$\frac{\partial V^1}{\partial m_1} = -\frac{\partial \pi_1}{\partial c_1}(m_1,c_2)F_1(m_1) > 0$$

and

$$\frac{\partial V^1}{\partial c_2} = \int_{\underline{c}}^{m_1}\left[\frac{\partial \pi_1}{\partial c_2}(c_1,c_2) - \frac{\partial \pi_1}{\partial c_2}(m_1,c_2)\right]dF_1(c_1) > 0$$

for all $(m_1,c_2) \in [\underline{c},\bar{c}]^2$ by A3.

Q.E.D.

As i's own best draw declines, the likelihood of getting increasingly better draws declines while the sampling cost is fixed. As the opponent's cost declines, the marginal value of a lower



production cost parameter for i declines since $\frac{\partial^2 \pi_i}{\partial c_i \partial c_j} < 0$. So in either case the marginal value (to i) of another trial declines. The sign of $V^1(\bar{c},\underline{c})$ is also of interest.

A4. (a) $V^1(\bar{c},\underline{c}) > 0$.

(b) $V^1(\bar{c},\underline{c}) \leq 0$.

If A4(a) holds, then since $\partial V^1/\partial c_2 > 0$, $V^1(\bar{c},c_2) > 0$ for all $c_2 \in [\underline{c},\bar{c}]$. If A4(b) holds, then $V^1(\bar{c},c_2) \leq 0$ for an interval $[\underline{c},c'] \subseteq [\underline{c},\bar{c}]$.

Which of these properties are passed through to $U^1(m_1,\xi_2)$? From expression (2), it is clear that for $(m_1,\xi_2) \in [\underline{c},\bar{c}] \times [\underline{c},\bar{c})$, $U^1(m_1,\xi_2)$ is continuously differentiable with

$$\frac{\partial U^1}{\partial m_1}(m_1,\xi_2) = \int_{\underline{c}}^{\xi_2} \frac{\partial V^1}{\partial m_1}(m_1,c_2)\, dF_2(c_2)/F_2(\xi_2) > 0. \tag{6}$$

Furthermore, using Leibniz' rule and collecting terms,

$$\frac{\partial U^1(m_1,\xi_2)}{\partial \xi_2} = \frac{dF_2(\xi_2)}{F_2(\xi_2)} \int_{\underline{c}}^{\xi_2}[V^1(m_1,\xi_2) - V^1(m_1,c_2)]\, dF_2(c_2)/F_2(\xi_2)>0 \tag{7}$$

where the inequality follows from the fact that $V^1$ is increasing in its second argument.

If A4(a) holds, then $U^1(\bar{c},\xi_2) > 0$ for all $\xi_2 \in [\underline{c},\bar{c}]$. If A4(b) holds, then $U^1(\bar{c},\xi_2) \leq 0$ for an interval $[\underline{c},c''] \subseteq [\underline{c},\bar{c}]$.

<u>Definition 3'</u>. A <u>best response function</u> for 1 is a function $\phi_1$: $[\underline{c},\bar{c}] \quad [\underline{c},\bar{c}]$ such that

$$\phi_1(\xi_2) = \max \{m_1 \varepsilon [\underline{c},\bar{c}] \mid U^1(m_1,\xi_2) \leq 0\}.$$

A strategy $\xi_1$ is a <u>best response</u> for 1 to $\xi_2$ if $\xi_1 = \phi_1(\xi_2)$.

A best response function for 2 is defined in the obvious way. The next step is to demonstrate the existence of and to describe the best response functions.

<u>Proposition 4</u>. There exists a nonincreasing best response function for i which may be discontinuous only at $\xi_j = \bar{c}$.

<u>Proof:</u> See the Appendix.

Proposition 4 implies that one can characterize the optimal contingent stopping rule for i as follows:

> given that j will stop whenever $m_j \leq \xi_j$ and continue otherwise, i's best response is to stop whenever $m_i \leq \phi_i(\xi_j)$; otherwise continue.

A best response function is illustrated in Figure 1. The fact that $\phi_i(\xi_j)$ is downward sloping implies that the more aggressive the the development strategy the rival j undertakes (the lower is $\xi_j$), the less aggressive is the optimal development strategy for i (the higher is $\phi_i(\xi_j)$).



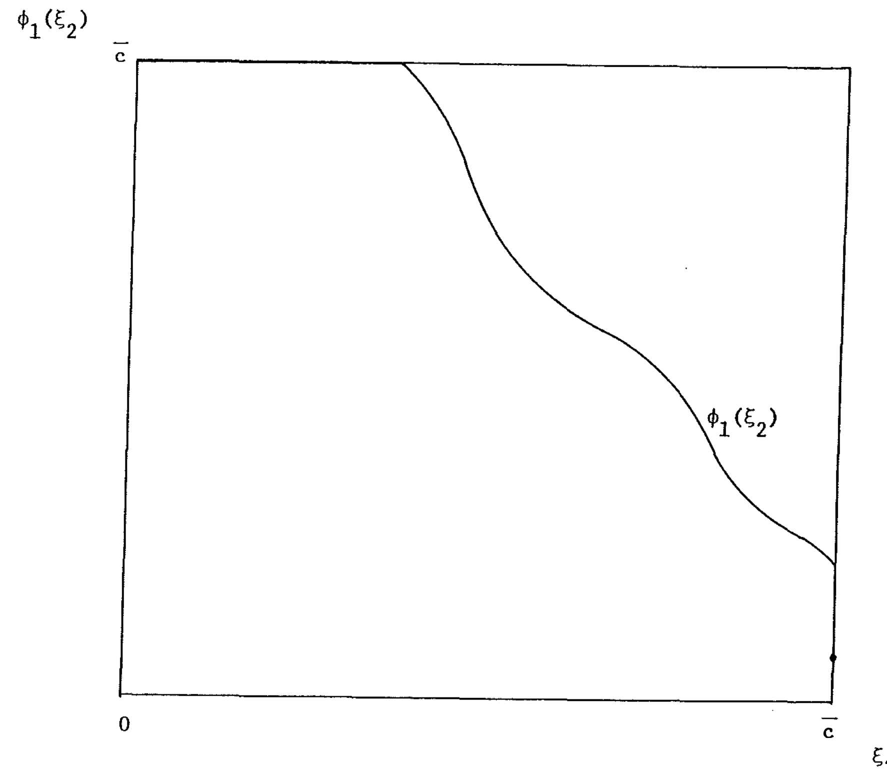


FIGURE 1

<u>Definition 4.</u> A strategy pair $(\xi_1^*, \xi_2^*)$ is a <u>Nash equilibrium</u> if $\xi_1^* = \phi_1(\xi_2^*)$ and $\xi_2^* = \phi_2(\xi_1^*)$.

<u>Proposition 5.</u> There exists a Nash equilibrium $(\xi_1^*, \xi_2^*)$ in $C_1 \times C_2$.

<u>Proof.</u> Define the function $\psi(\xi_1) = \phi_1 \circ \phi_2 : [\underline{c}, \bar{c}] \to [\underline{c}, \bar{c}]$. $\psi(\xi_1)$ is nondecreasing and is therefore upper semicontinuous from the left and lower semicontinuous from the right. Application of Roberts' and Sonnenschein's [1976] lemma implies that the function $\psi(\xi_1)$ has a fixed point $\xi_1^*$. Then $\xi_2^* = \phi_2(\xi_1^*)$ completes the Nash equilibrium pair $(\xi_1^*, \xi_2^*)$.

Q.E.D.

## IV. COMPARATIVE STATICS

The usual comparative statics results can be demonstrated to apply to the best response functions.

<u>Proposition 6</u>. Whenever $\phi_i(\xi_j) < \bar{c}$

(a) $\dfrac{\partial \phi_i(\xi_j)}{\partial k_i} > 0$

and

(b) a mean profit-preserving spread of $F_i$ (in the sense of Diamond and Stiglitz [1974]) decreases $\phi_i(\xi_j)$.

Proof: See the Appendix.



Thus for any fixed $\xi_j$, an increase in i's own development costs results in less development, while a mean profit-preserving spread results in increased development activity.

However, these intuitive results need not extend to the equilibrium reservation cost levels $(\xi_1^*, \xi_2^*)$. To see this consider the example in Figure 2, in which there are three equilibria: A, B, and C. One would intuitively expect that $\partial\xi_1^*/\partial k_1 > 0$, $\partial\xi_1^*/\partial k_2 < 0$, $\partial\xi_2^*/\partial k_1 < 0$, and $\partial\xi_2^*/\partial k_2 > 0$. While this is true at A and C, the signs are completely reversed at B. To see this, recall that $\partial\phi_1(\xi_2)/\partial k_1 > 0$ whenever $\phi_1(\xi_2) \in (\underline{c}, \bar{c})$. This means that at Nash equilibrium B, $\partial\xi_1^*/\partial k_1 < 0$ and $\partial\xi_2^*/\partial k_1 > 0$. That is, an increase in firm 1's development cost induces 1 to undertake <u>more</u> development activity, while its rival reduces its development activity (see Figure 3). Similarly, since a mean profit-preserving spread in $F_1$ lowers the function $\phi_1(\xi_2)$, this spread actually increases $\xi_1^*$ (and decreases $\xi_2^*$) at B (see Figure 4).

A related question is that of the effect of asymmetry. The possibility of multiple equilibria makes it clear that the firm with higher development costs <u>need not</u> have a higher equilibrium reservation cost than its lower-development cost rival. Similarly, firms with identical development costs need not end up at a symmetric Nash equilibrium in reservation costs.

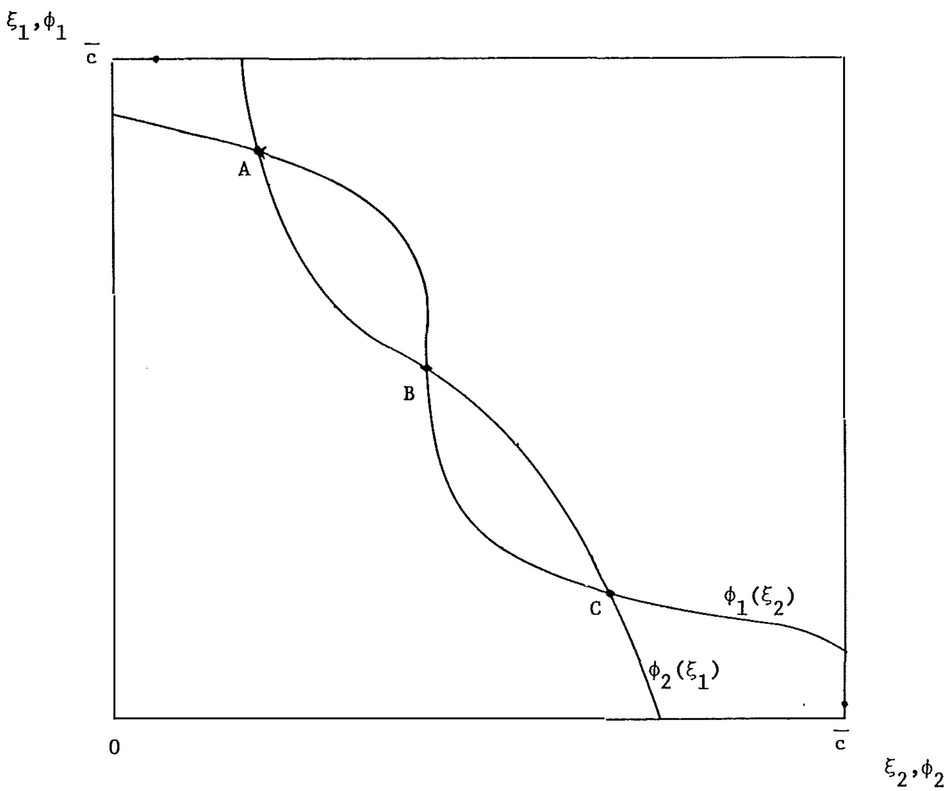


FIGURE 2



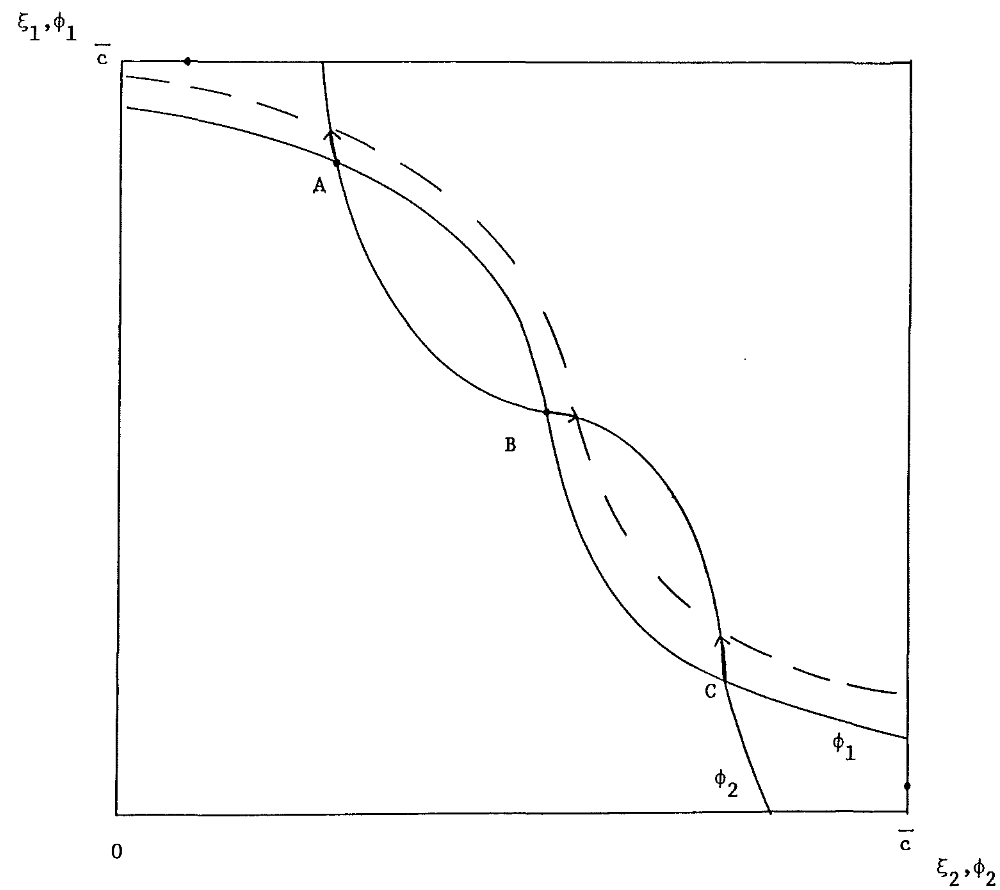


FIGURE 3

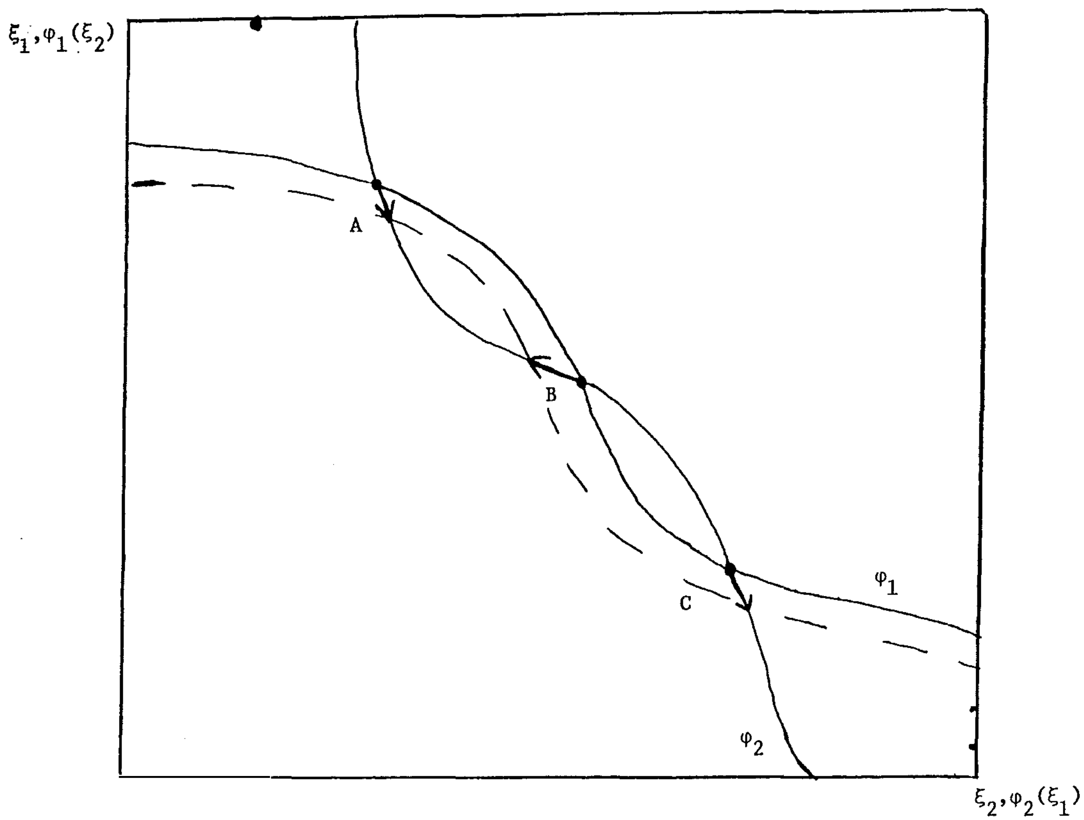


FIGURE 4



## V. EFFECTS OF RIVALRY

An interesting question is whether a particular firm will engage in more development activity when faced by a rival or when that firm is a monopolist.[5]

Let us designate firm 1 as our particular firm. Then $\pi(c_1)$ denotes monopoly profit when marginal production cost is $c_1 \varepsilon [\underline{c},\bar{c}]$, while $\pi_1(c_1,c_2)$ denotes firm 1's profits when faced by a rival with marginal production cost of $c_2$. It is quite reasonable to claim that $\pi(c_1) \geq \pi_1(c_1,c_2)$ for all $(c_1,c_2) \varepsilon [\underline{c},\bar{c}]^2$. Unfortunately, the critical expression is not the <u>level</u> of profits, but the <u>increment</u> to profits generated by an additional development trial. Again, let $m_1$ denote 1's lowest cost to date and let $\hat{\xi}_1$ represent firm 1's (monopolistic) reservation cost level.

<u>Proposition 7</u>. If $\pi(c_1) - \pi(m_1) \geq \pi_1(c_1,\bar{c}) - \pi_1(m_1,\bar{c})$ for all $(m_1,c_1) \varepsilon [\bar{c},\bar{c}]^2$ with $m_1 \geq c_1$, then $\hat{\xi}_1 \leq \xi_1^*$, where $(\xi_1^*,\xi_2^*)$ is any Nash equilibrium pair.

That is, if the increment to firm 1's monopoly profits (due to a cost decrease from $m_1$ to $c_1$) exceeds the increment to firm 1's profits when faced with a rival having a production cost $\bar{c}$, then firm 1 will choose a lower reservation cost level as a monopolist than as a duopolist. Intuitively, this is because a monopolist, having a greater volume of sales than a duopolist, benefits more from a given cost reduction than does a duopolist (under the hypothesis of the proposition). Hence a monopolist may find continued development profitable where a duopolist will have ceased.

<u>Proof:</u> Since $\phi_1(\xi_2)$ is nonincreasing one need only show that $\hat{\xi}_1 \leq \phi_1(\bar{c})$. Then $\hat{\xi}_1 \leq \phi_1(\xi_2)$ for all $\xi_2$ and since $\xi_1^* = \phi(\xi_2^*)$ at a Nash equilibrium, $\hat{\xi}_1 \leq \xi_1^*$. Inequality (1) implies that

$$0 \geq \int_{\underline{c}}^{\hat{\xi}_1} [\pi(c_1) - \pi(\hat{\xi}_1)]dF_1(c_1) - k_1$$

$$\geq \int_{\underline{c}}^{\hat{\xi}_1} [\pi_1(c_1,\bar{c}) - \pi_1(\hat{\xi}_1,\bar{c})]dF_1(c_1) - k_1 = U^1(\hat{\xi}_1,\bar{c})$$

where the second inequality follows from the hypothesis of the proposition.

Now $\phi_1(\bar{c})$ is either $\bar{c}$ or is uniquely defined by $U^1(\phi_1(\bar{c}),\bar{c}) = 0$. Since $U^1$ is increasing in its first argument, in either case we have $\hat{\xi}_1 \leq \phi_1(\bar{c})$. The proposition follows. Of course, an analogous proposition holds for firm 2.

Q.E.D.

It would be nice to know all the circumstances under which the hypothesis of Proposition 7 holds. While it is impossible to tell in general, for the case of the linear demand curve P = a - bQ, a sufficient condition is[6] $a - \underline{c} \geq 8(\bar{c} - \underline{c})$. Since the hypothesis also represents a sufficient, but not necessary condition, the results of Proposition 7 hold somewhat more generally.



For any choice of reservation cost level $\xi$, the number of development trials required is a geometrically-distributed random variable with mean $1/F_1(\xi)$. Thus firm 1's expected costs as a monopolist are $k_1/F_1(\hat{\xi}_1)$, while firm 1's expected costs as a duopolist are $k_1/F_1(\xi_1^*)$. Since $\hat{\xi}_1 \leq \xi_1^*$, firm 1 spends more as a monopolist than as a duopolist. However, total duopoly costs are $[k_1/F_1(\xi_1^*) + k_2/F_2(\xi_2^*)]$. Thus one cannot say with any confidence whether monopoly development costs exceed or fall short of total duopoly costs.

An alternative interpretation of nonrivalrous behavior is that of a cartel, rather than a monopoly. That is, suppose firms 1 and 2 enter into an agreement to coordinate their development strategies so as to maximize joint profits. What is an optimal development strategy in this case? Since marginal costs have been assumed constant, only the minimum cost firm will produce output in the new market. Thus the joint payoff will be <u>either</u> $\pi(c_1)$ or $\pi(c_2)$ where $\pi(c_i)$ represents monopoly profits when firm i supplies the good at cost $c_i$.

Suppose that at each stage the cartel may elect to conduct a development trial for firm 1 or for firm 2, but not for both simultaneously. Because there is essentially instantaneous sampling, this is without loss of generality. Then application[7] of Pandora's Rule (Weitzman, 1979) yields the following optimal strategy.

First compute $\hat{\xi}_j = \min\{\hat{\xi}_1,\hat{\xi}_2\}$ where $\hat{\xi}_i$, i = 1,2 is the reservation cost level for firm i if i were a monopolist. Then firm j should conduct development trials until a production cost of $c_j \leq \hat{\xi}_j$ is drawn; then stop. No development trials should be undertaken by firm $i \neq j$.

Thus the cartel will engage in more development activity than the noncooperative firms if the hypothesis of Proposition 7 holds for either firm 1 or firm 2.

## VI. CONCLUSIONS

This paper has shown that traditional search theory can be extended in a natural way to include game-playing agents. The existence of Nash equilibria in stopping rules is demonstrated. Furthermore, the equilibrium strategies are shown to be myopic and possess the reservation property. However, the existence of multiple equilibria can provide some counterintuitive results regarding the effects of increases in development costs on the equilibrium reservation production cost levels. For example, an increase in firm i's development costs can cause firm i to undertake <u>more</u> development.

It seems clear that this analysis could be extended to the case of n firms. However, since the expressions like $U^1(m_1,\xi_2,\xi_3,\ldots,\xi_n)$ will now have a jump whenever $\xi_j = \bar{c}$ $(j \neq 1)$, there will be $(n-1)!$ such discontinuities for each player (and the analyst!) to contend with. Several other directions for generalization are obvious. Discounting, allowing simultaneous production and development, and allowing rivals to observe the results of development trials are but a



few. One would suspect that the results of this paper on the optimality of myopic stopping rules would be particularly sensitive to changes in the assumptions regarding the observability of the rival's development progress.

## APPENDIX

Proof of Proposition 4. For $i = 1$. The proof for $i = 2$ is analogous.

Consider $\xi_2 \in [\underline{c},\bar{c})$. If A4(a) holds, then $U^1(\bar{c},\xi_2) > 0$ for all $\xi_2 \in [\underline{c},\bar{c})$. Since $U^1(\underline{c},\xi_2) = -k_1 < 0$ for all $\xi_2 \in [\underline{c},\bar{c})$ and since $U^1(m_1,\xi_2)$ is continuously differentiable in $m_1$ for each fixed $\xi_2$ with $\partial U^1/\partial m_1 > 0$, the intermediate value theorem and monotonicity guarantee the existence of a unique value $\xi_1 \in [\underline{c},\bar{c})$ such that $U^1(\xi_1,\xi_2) = 0$. For $\xi_2 \in [\underline{c},\bar{c})$, $U^1$ is continuously differentiable in $(m_1,\xi_2)$ with $\partial U^1/\partial m_1 > 0$, so there exists an implicit function $\phi_1(\xi_2)$ such that $U^1(\phi_1(\xi_2),\xi_2) = 0$ for all $\xi_2 \in [\underline{c},\bar{c})$. Furthermore, $\phi_1(\xi_2)$ is continuously differentiable with

$$\phi_1'(\xi_2) = -\frac{\partial U^1(\phi_1(\xi_2),\xi_2)/\partial \xi_2}{\partial U^1(\phi_1(\xi_2),\xi_2)/\partial m_1}.$$

By inequalities (6) and (7), $\phi_1'(\xi_2) < 0$ for all $\xi_2 \in [\underline{c},\bar{c})$ whenever A4(a) holds. If A4(b) holds, then there exists an interval (possibly $[\underline{c},\bar{c})$) where $U^1(\bar{c},\xi_2) \leq 0$. For all $\xi_2$ in this interval, $\phi_1(\xi_2) \quad \bar{c}$.

The continuity of $U^1(m_1,\xi_2)$ on $[\underline{c},\bar{c}] \times [\underline{c},\bar{c})$ ensures the continuity of $\phi_1(\xi_2)$ on $[\underline{c},\bar{c})$.

However, $\phi_1(\xi_2)$ may take a downward jump at $\xi_2 = \bar{c}$. To see this, recall that $U^1(m_1,\bar{c}) = V^1(m_1,\bar{c})$. If $V^1(\bar{c},\bar{c}) \leq 0$, then $\phi_1(\bar{c}) = \bar{c}$. But



$V^1(\bar{c},\bar{c}) \leq 0$ implies that $U^1(m_1,\xi_2) \leq 0$ for all $(m_1,\xi_2) \in [\underline{c},\bar{c}]^2$. So $\phi_1(\xi_2) \equiv \bar{c}$ in this case. If $V^1(\bar{c},\bar{c}) > 0$, then since $V^1(\underline{c},\bar{c}) < 0$ and $V^1$ is increasing in its first argument, $\phi_1(\bar{c})$ lies strictly between $\underline{c}$ and $\bar{c}$ and is uniquely defined by $V^1(m_1,\bar{c}) = 0$, while $\hat{\phi}_1(\bar{c}) = \lim_{\xi_2 \to \bar{c}_-} \phi_1(\xi_2)$ is defined by

$$\hat{\phi}_1(\bar{c}) = \bar{c} \text{ if } \lim_{\xi_2 \to \bar{c}_-} U^1(\bar{c},\xi_2) \leq 0 \tag{8}$$

and $\hat{\phi}_1(\bar{c})$ is implicitly defined by

$$\lim_{\xi_2 \to \bar{c}_-} U^1(\hat{\phi}_1(\bar{c}),\xi_2) = 0 \text{ if } \lim_{\xi_2 \to \bar{c}_-} U^1(\bar{c},\xi_2) > 0. \tag{9}$$

Since $\bar{c} > \phi_1(\bar{c}) \in (\underline{c},\bar{c})$, case (8) is done.

Consider case (9). Because $V^1(m_1,\bar{c}) > V^1(m_1,c_2)$ for all $c_2 < \bar{c}$,

$$V^1(m_1,\bar{c}) > \int_{\underline{c}}^{\xi_2} V^1(m_1,c_2)dF_2(c_2)/F_2(\xi_2) \text{ for all } \xi_2 \in [\underline{c},\bar{c}].$$

Thus $V^1(m_1,\bar{c}) > \int_{\underline{c}}^{\bar{c}} V^1(m_1,c_2)dF_2(c_2)/F_2(\bar{c}) = \lim_{\xi_2 \to \bar{c}_-} U^1(m_1,\xi_2)$.

Since these inequalities hold for all $m_1 \in [\underline{c},\bar{c}]$,

$$U^1(\hat{\phi}_1(\bar{c}),\bar{c}) = V^1(\hat{\phi}_1(\bar{c}),\bar{c}) > \lim_{\xi_2 \to \bar{c}_-} U^1(\hat{\phi}_1(\bar{c}),\xi_2) = 0$$

implying that $\hat{\phi}_1(\bar{c}) > \phi_1(\bar{c})$.

(Q.E.D.)

Proof of Proposition 6.

(a) $$\frac{\partial\phi_1(\xi_2)}{\partial k_1} = \frac{-\partial U^1(\phi_1(\xi_2),\xi_2)/\partial k_1}{\partial U^1(\phi_1(\xi_2),\xi_2)/\partial m_1} > 0$$

in view of (6). Similarly for $i = 2$.

(b) Following Kohn and Shavell, let $z$ be a parameter of $F_1$, whose increase represents an increase in riskiness. However, $F_1(c_1;z)$ is required to satisfy the analogs of the Diamond-Stiglitz conditions (12) and (13) for a mean utility-preserving spread. For fixed $\xi_2$,

$$\int_{\underline{c}}^{m_1}\int_{\underline{c}}^{\xi_2}\frac{\partial\pi_1}{\partial c_1}(c_1,c_2)[dF_2(c_2)/F_2(\xi_2)]\ \frac{\partial F_1}{\partial z}(c_1;z)dc_1 \leq 0$$

for all $m_1$, with equality when $m_1 = \bar{c}$.[8]

The expected benefits of sampling become

$$U^1(m_1,\xi_2;z) = \int_{\underline{c}}^{\xi_2}\left[\int_{\underline{c}}^{m_1}\{\pi_1(c_1,c_2) - \pi_1(m_1,c_2)\}dF_1(c_1;z) - k_1\right]dF_2(c_2)/F_2(\xi_2).$$

Since $\dfrac{\partial\phi_1(\xi_2)}{\partial z} = \dfrac{-\partial U^1(\phi_1(\xi_2),\xi_2;z)/\partial z}{\partial U^1(\phi_1(\xi_2),\xi_2;z)/\partial m}$, in view of (6) we are interested in sgn $\partial U^1/\partial z$.



$$\frac{\partial U^1}{\partial z} = \int_{\underline{c}}^{\xi_2}\int_{\underline{c}}^{m_1}\{\pi_1(c_1,c_2) - \pi_1(m_1,c_2)\}\ \frac{\partial[dF_1(c_1;z)]}{\partial z}\ dF_2(c_2)/F_2(\xi_2).$$

Integration by parts with respect to $c_1$ yields

$$\frac{\partial U^1}{\partial z} = \int_{\underline{c}}^{\xi_2}\{\pi_1(c_1,c_2) - \pi_1(m_1,c_2)\}\ \frac{\partial F_1}{\partial z}(c_1;z)\Big|_{c_1=\underline{c}}^{m_1} dF_2(c_2)/F_2(\xi_2)$$

$$-\int_{\underline{c}}^{m_1}\int_{\underline{c}}^{\xi_2}\frac{\partial\pi_1(c_1,c_2)}{\partial c_1}[dF_2(c_2)/F_2(\xi_2)]\ \frac{\partial F_1}{\partial z}(c_1,z)dc_1.$$

The first expression is zero since $\partial F_1(\underline{c};z)/\partial z = 0$ (i.e., $F_1(\underline{c};z) = 0$ for all $z$). The second expression is nonnegative by (10). Therefore $\partial\phi_1(\xi_2)/\partial z \leq 0$. The proof for $i = 2$ is analogous.

(Q.E.D.)

## FOOTNOTES

1. Jensen [1980] is in the process of generalizing his earlier work [1979] to the duopolistic case. His work differs analytically from this paper, but the two are in the same spirit.

2. The payoffs $\pi_1(c_1,c_2)$, $\pi_2(c_1,c_2)$ are envisioned as the allocations generated by Nash equilibrium output levels $(q_1^e,q_2^e)$ in the duopoly game with payoffs

$$P_1(q_1,q_2) = \frac{1}{r}\{p(q_1 + q_2)q_1 - c_1q_1\}$$

$$P_2(q_1,q_2) = \frac{1}{r}\{p(q_1 + q_2)q_2 - c_2q_2\}.$$

3. These assmptions are easily shown to be valid for a number of simple demand functions. Examples include $P = a - b\ln Q$, $P = a - bQ$, $P = a + b/Q$.

4. Sufficient conditions for the equilibrium to be unique are A4(a), an analogous condition on $V^2(\underline{c},\bar{c})$, and

$$\partial U^1(\phi_1(\xi_2),\xi_2)/\partial m_1 > \partial U^1(\phi_1(\xi_2),\xi_2)/\partial\xi_2$$

and

$$\partial U^2(\xi_1,\phi_2(\xi_1))/\partial m_2 > \partial U^2(\xi_1,\phi_2(\xi_1))/\partial\xi_1.$$



In this case, $\phi_i(\xi_j) \in (\underline{c},\bar{c})$ and $-1 < \phi_i'(\xi_j) < 0$, so that a unique equilibrium exists.

5. I am indebted to John Roberts for suggesting this section.

6. For $P = a - bQ$, $\pi(c_1) - \pi(m_1) \geq \pi_1(c_1,\bar{c}) - \pi_1(m_1,\bar{c})$ for all $m_1$ and $c_1$ with $m_1 \geq c_1$ if and only if $2a - 16\bar{c} + 7m_1 + 7c_1 \geq 0$ for all $m_1$ and $c_1$ with $m_1 \geq c_1$. A sufficient condition for this to be true is $a - \underline{c} \geq 8(\bar{c} - \underline{c})$. Note that this is independent of the slope coefficient b.

7. I am indebted to Carl J. Lydick for pointing out the applicability of Pandora's Rule to this case.

8. The sign reversal is due to the fact that Diamond and Stiglitz use utility, which is increasing in its argument, while i's profit decreases in $c_i$.

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